

Axiomatization of Mechanics

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The problem of axiomatization of physics formulated by Hilbert as early as 1900 and known as the Sixth Problem of Hilbert is nowadays even more topical than at the moment of its formulation. Axiomatic inconsistency of classic, quantum, and geometrized relativistic physics of the general relativistic theory does not in the least fade away, but on the contrary, becomes more pronounced each year. This naturally evokes the following questions: 1. Is it possible, without drastically changing the mathematics apparatus, to set up the axiomatics of physics so as to transform physics, being presently a multitude of unmatched theories with inconsistent axiomatics, into an integrated science? 2. Is it possible, maybe through expanding their scopes, to generalize or transform the existing axiomatics into an integral system of axioms in such a manner that existing axiomatics of inconsistent theories would follow there from as a particular case?

Keywords: axiomatization, kinematic state, Mach principle, Kinematic Principle, Dynamic Principle, Static Principle.

PACS: 45.20.Jj, 45.50.Dd, 03.65.Ud.

An axiom is a statement adopted without proof. Therefore, there is no other way of obtaining an axiom than to merely guess it. It is rather hard to resist the temptation of declaring an axiom every phenomenon inexplicable in the framework of a particular theory. In such a case, on the one hand, the consistency of a theory would not be compromised, and on the other hand, the problem of explanation of this phenomenon is avoided. However, this approach is erroneous and fallacious, as the number of axioms cannot grow in an uncontrollable manner; the number of axioms shall be minimized. Only this way one can expect minimization of likely errors should some axioms be guessed inadequately. At the same time, it is impossible to avoid axioms at all, because, according to Godel theorem, each theory comprises statements impossible to be proved within the framework of this theory. It is these statements that constitute the foundation of the theory, governing its results and implications.

On the one hand, it seems evident that having altered a single word in an axiom, we could obtain dramatic changes in the theory. Hence, arbitrary altering of axioms is inadmissible, as otherwise we would get a chaotic set of axioms rather than an axiomatic system. On the other hand, science is not a church doctrine, but rather, a system of theories based on guessed axioms. Therefore, one should be extremely careful in altering axioms to adapt them to the obtained new results. Otherwise, the description of the physical reality would result as a multitude of inconsistent, often mutually contradicting, axioms. And, as it has been mentioned, the number of axioms should be reduced to the minimum.

Historically, the axiomatics of physics has by all means experienced alterations. For example, if we compare the classical Newtonian physics with its predecessor, Aristotelian physics, we can easily see that while axiomatics of Aristotelian physics presumed motion to occur only provided a force being ap-

plied to the body, the axiomatics of Newtonian classical physics states that motion may occur also if no force is applied to the body. Therefore, Aristotelian physics postulates that the dynamics of bodies is described by first order differential equations, whereas Newtonian physics (with the Laws of Newton being from the mathematical viewpoint the axiomatics of classical physics[1]) postulates proportionality of the force applied to the body to its acceleration. In other words, the Laws of Newton postulate the description of body dynamics by the second order differential equations. Newton employed the concept of the absolute space being the space related to fixed stars. Such space can be called Euclidian.

Is it correct to consider the Laws of Newton to be the axiomatics of classical physics? The answer is definitely positive. Shall we, and may we, expand their scope of application to microobjects? Quantum mechanics is a physical description of particles employing the definition of inertial reference frame, hence employing the First Newton's Law. The First Law states that any body free from interactions with other bodies would have constant velocity. So, how is velocity defined in quantum mechanics? It is done through the average with wave function ψ

$$\langle \dot{x} \rangle = \int \psi^* x \psi dx. \quad (1)$$

In a quantum (real) reference frame $\langle \dot{x} \rangle = \text{const}$ there always exist infinitesimal fields, waves, and forces perturbing an ideal inertial reference frame. This follows from one of the general definitions of Mach principle [2]: "Local physical laws are determined by large-scale structure of the Universe". Commenting the Mach principle, let us note that in this case the definition of inertial properties of a body is determined by multi-particle interactions with all bodies in the Universe. The description of the case of particle motion with higher derivatives of coordinates in time has been for the first time published in 1850 by M. Ostrogradsky; it is known as an Ostrogradsky Canonical Formalism [3]. Being a mathematician, Ostrogradsky considered coordinate systems rather than reference systems. This case corresponds to a quantum (real) reference frame comprising not only inertial reference frames, but also non-inertial ones, determined, according to Mach principle, by multi-particle interactions with all bodies moving in the Universe. Inertial reference frames defines by all bodies moving in the Universe [2]

$$\frac{d^2}{dt^2} \left(\frac{\int \rho(r) r dV}{\int \rho(r) dV} \right) = 0.$$

Here $\rho(r)$ is the function of the distribution of all masses in the Universe with the volume V .

Definition

A kinematic state of a mechanical system with constant higher derivative $\dot{x}^{(n)} = \text{const}$ is called defined if the kinematics of the body is described with a differential equation

$$F(x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}, \dots, \dot{x}^{(n)}) = 0 \quad (2)$$

Let us assume that if in an arbitrary (any) reference frame the average value of a higher derivative is constant,

$$\langle \dot{x}^{(n)} \rangle = \langle \frac{d^n x}{dt^n} \rangle = \text{const} \quad (3)$$

then the function F is finite.

Kinematic Principle (Inertial Principle)

A kinematic state of a mechanical system free from interactions with other bodies is observer-dependent and persists until its interaction with other bodies alters its kinematic state.

The acceleration for a body with free from interactions with other bodies is a constant for the observer in the constant-accelerated reference frame. In this case the acceleration is define the kinematic state of the body because in that case the acceleration is the invariant for the reference frame of the observer.

Let us call an invariant of a reference frame the constant higher derivative that does not change in case of transformation of coordinates

$$x' = f(x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}, \dots, \dot{x}^{(n)}) \quad (4)$$

$$t' = t \quad (5)$$

Then the kinematic state of a mechanical system free form interactions with other bodies depends on the invariant of the observer's reference frame.

Let us call a harmonic reference system the reference system with a clock and an observer oscillating harmonically, in which any body free from interactions with other bodies would maintain the average value of its higher derivative. For a reference system oscillating harmonically let us consider the invariant the $\langle \dot{x}^{(n)} \rangle = \text{const}$. In a harmonic reference frame a coordinate of the body may be described by the function

$$\varphi(t, x) = \varphi_0 \exp i(kx + \omega t) \quad (6)$$

being φ_0 the amplitude of oscillation, k and ω the wave vector and angular frequency of oscillations, respectively.

In the particular case,

$$\varphi(t, x) \approx kx + \omega t = K_i X^i \quad (7)$$

and being K_i 4-dimensional wave vector and X^i 4-coordinate of the body, $i = 0, 1, 2, 3$ in a harmonic reference frame. In this case, function φ describes the 4-coordinate of the body multiplied by a constant coefficient. In a vibrating reference frame the coordinate of a body may be expressed with arrays of harmonic oscillations.

Dynamic Principle

There exist reference frames with a clock and an observer, in which the dynamics of a body is described by the equation:

$$k_1\dot{x} + k_2\ddot{x} + k_3\dddot{x} + \dots k_{2n}\dot{x}^{(2n)} = F(x, \dot{x}, \ddot{x}, \dddot{x}, \dots, \dot{x}^{(n)}) \quad (8)$$

We will call such reference frames real (or quantum reference frames).

Let us call a force F the quantitative measure of the interaction between the bodies.

The generalized principle of relativity of Galileo means in this case that the order of the differential equation (6) describing the dynamics of a mechanical system with the invariant of the reference frame $\dot{x}^{(n)} = \text{const}$ does not alter at transformation (3).

The differential equation (6) corresponds to the description of the body dynamics in a non-isolated (open) mechanical system with the external forces of the system with odd derivatives, corresponding, for example, to losses due to friction and radiation. Odd derivatives correspond to losses (friction or radiation) and describe irreversible cases for open systems not satisfying variational principles of mechanics. The case of an isolated (close) mechanical system corresponds the differential equation with even derivatives.

$$k_2\ddot{x} + k_4\ddot{\ddot{x}} + \dots k_{2n}\dot{x}^{(2n)} = F(x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}, \dots, \dot{x}^{(n)}) \quad (9)$$

The reference frames, in which the dynamics of a system is described by the equation

$$< k_2\ddot{x} > = < F(x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}, \dots, \dot{x}^{(n)}) >, \quad (10)$$

we will call inertial reference frames. Here, the proportionality coefficient in the equation k_2 of dynamics (6) is the mass of a body.

Static Principle

If a particle rests along an arbitrary direction, then the resultant force acting thereon along this direction is zero.

For inertial frames, the Lagrangian L depends only on coordinates and their first derivatives $L = L(x, \dot{x})$ [4]. For the case of quantum (real) reference frames the Lagrangian depends on coordinates and their higher derivatives and has the form $L = L(x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}, \dots, \dot{x}^{(n)})$.

Let us consider in more detail such precise description of dynamics of bodies motion accounting for quantum (real) reference frames determined, according to our model, by complex multi-particle interactions with all bodies in the Universe.

For an accurate description of dynamics of bodies motion accounting for higher derivatives, let us consider the body in an arbitrary reference frame, denoting the position r of the body in the space as and time as t . Then, expanding the function $r = r(t)$ into Taylor's series in the zero point, we get

$$r = r_0 + \dot{r}t + \frac{1}{2!}\ddot{r}t^2 + \frac{1}{3!}\ddot{\ddot{r}}t^3 + \dots + \frac{1}{n!}\dot{r}^{(n)}t^n + \dots \quad (11)$$

Let us denote

$$r_N = r_0 + \dot{r}t + \frac{1}{2!}\ddot{r}t^2$$

and the additional correction variables q_r for our model with arbitrary reference frames as , where - is a value equal to zero in the classical Newtonian mechanics

$$q_r = \frac{1}{3!}\ddot{r}t^3 + \dots + \frac{1}{n!}\dot{r}^{(n)}t^n + \dots \quad (12)$$

Then

$$r = r_N + q_r.$$

In our case the description discrepancy and *uncertainty between the two models* h is equal to the difference in descriptions of a test particle in the extended Newtonian dynamics with the Lagrangian $L = L(x, \dot{x}, \ddot{x}, \dots, \dot{x}^{(n)})$ and Newtonian dynamics in the inertial reference frames with the Lagrangian $L = L(x, \dot{x})$:

$$\int (L(x, \dot{x}, \ddot{x}, \dots, \dot{x}^{(n)}) - L(x, \dot{x}))dt = S(x, \dot{x}, \ddot{x}, \dots, \dot{x}^{(n)}) - S(x, \dot{x}) = h \quad (13)$$

Let us apply the least action principle [5]:

$$\delta S = \delta \int L(\dot{r}', r')dt = \int \sum_{n=0}^N (-1)^n \frac{d^n}{dt^n} \frac{\partial L}{\partial \dot{r}^{(n)}} \delta \dot{r}^{(n)} dt = 0. \quad (14)$$

Then the generalized Euler-Lagrange equation for real reference frames will take on the form

$$\sum_{n=0}^N (-1)^N \frac{d^N}{dt^N} \frac{\partial L}{\partial \dot{r}^{(N)}} = 0. \quad (15)$$

Or,

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{r}} - \dots + (-1)^N \frac{d^N}{dt^N} \frac{\partial L}{\partial \dot{r}^{(N)}} = 0. \quad (16)$$

Let us generalize the above for the case of curvilinear coordinates. To do so, one has to take into account the fact that in case of parallel translation of a vector along non-straight trajectory of the body, not only its value could be altered in the curved space, but as well its direction. Therefore, applying covariant derivative in parameter τ of the vector $A^a, a = 1, 2, 3$

$$\nabla_i A^k = \frac{\partial A^k}{\partial x^i} + \Gamma_{ij}^k A^j$$

Let us introduce the operator

$$\begin{aligned} \dot{D}^{(1)} &= \frac{dx^i}{d\tau} \nabla_i = \frac{D}{D\tau} \\ \dot{D}^{(2)} A^i &= \frac{\partial \dot{D}^{(1)} A^i}{\partial \tau} + \Gamma_{jk}^i \dot{D}^{(1)} A^j \frac{dx^k}{d\tau} \\ \dot{D}^{(N)} A^i &= \frac{\partial \dot{D}^{(N-1)} A^i}{\partial \tau} + \Gamma_{jk}^i \dot{D}^{(N-1)} A^j \frac{dx^k}{d\tau} \end{aligned}$$

Then the generalized Euler-Lagrange equations will take on the form

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{D}^{(1)}_r} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{D}^{(2)}_r} - \dots + (-1)^N \frac{d^N}{dt^N} \frac{\partial L}{\partial D^{(N)}_r} = 0$$

When we develop the proposed model, we have to employ functions implementing stochastic variables. In particular, this is necessary to consider the stochastic phase of oscillation that may be caused by stochastic fields. As the nature of these fields is unknown, let us illustrate this case on the example of a physical model with a stochastic gravitational/inertial background (for non-inertial reference frames) and a distribution function assuming uniform distribution of these fields in time and space. This means that we suggest illustrating fluctuations of gravitational/inertial fields and waves mathematically expressed by a stochastic curved space.

Then, considering quantum microobjects in the curved space, we must take into account the fact that the scalar product of two 4-vectors A^i and B^k is $g_{ik}A^iB^k$, where for weak gravitational fields one may use the value h_{ik} , which is the solution of Einstein's equations for the case of weak gravitational field in harmonic coordinates.

The correlation factor M of the projection of stochastic vector variables λ^i onto directions A^k and B^n set by the polarizers (all these vectors being unity ones) is [6]

$$|M| = |\langle AB \rangle| = |\langle \lambda^i A^k g_{ik} \lambda^m B^n g_{mn} \rangle| = \left| \frac{1}{2\pi} \int \cos \phi \cos(\phi + \theta) d\phi \right| = |\cos \theta|$$

due to the equations following from differential geometry,

$$\cos \phi = \frac{g_{ik} \lambda^i A^k}{\sqrt{\lambda^i \lambda_i} \sqrt{A^k A_k}},$$

$$\cos(\phi + \theta) = \frac{g_{mn} \lambda^m A^n}{\sqrt{\lambda^m \lambda_m} \sqrt{B^n B_n}}.$$

Here ϕ is the angle between λ^i and A^k , $(\phi + \theta)$ is between λ^m and B^n .

This means coincidence of the Bell's observable [7] with the experimental results in real (quantum) reference frames. All vectors here being unity ones with metrics averaging in the weak field approximation yielding unity; is the angle between polarizers, vector A^n is equal to vector B^n rotated by the angle θ , and indices taking the values $i = 0, 1, 2, 3$. Finally, we get

$$|M_{AB}| = |\cos \theta|. \quad (17)$$

The maximum value of the Bell's observable S is

$$|\langle S \rangle| = \frac{1}{2} |\langle M_{AB} \rangle + \langle M_{A'B} \rangle + \langle M_{AB'} \rangle - \langle M_{A'B'} \rangle| = \frac{1}{2} |\cos(-\frac{\pi}{4}) + \cos(\frac{\pi}{4}) + \cos(\frac{\pi}{4}) - \cos(\frac{3\pi}{4})| = \sqrt{2},$$

where $\theta = \frac{\pi}{4}$, being the doubled angle between direction of the polarizers A and B .

To sum it up, all equations of classical mechanics in the proposed model possess additional terms in the form of higher derivatives. At that, these additional terms are zero not always, but only in special cases, i.e. in the inertial reference frames.

Additional terms in the form of higher derivatives may play the role of hidden variables complementing both quantum and classic mechanics. Additional terms have non-local character, which enables their employment for description of non-local effects of quantum mechanics.

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